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Mass Transfer in Laminar Flow With Linear Wall Resistance—Entrance Region Solutions

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Numerous workers (Colton et al., 1971; Kooijman, 1973; Davis, 1973; Cooney et al., 1974; Walker and Davies, 1974) have analyzed the problem in which a fully developed laminar flow, between parallel plates or in a tube, encounters a mass transfer section where the flux to the wall is proportional to the wall concentration. For constant properties, low transfer rates, and negligible axial diffusion, the concentration profile is governed by

$$v_\xi^*(\eta)\,\frac{\partial \theta}{\partial x} = (\nabla^2\theta)_\eta \tag{1}$$

with

$$\theta = 1 \quad @ \quad x \leqq 0, \quad \text{all } \eta \tag{2}$$

$$\frac{\partial \theta}{\partial \eta} = 0 \quad @ \quad \eta = 0, \quad \text{all } x \tag{3}$$

and

$$\theta = -\epsilon \frac{\partial \theta}{\partial \eta} \quad @ \quad \eta = \pm 1, \quad x \geqq 0 \tag{4}$$

where

$$x = \frac{(\xi/h)}{Pe} \tag{5}$$

The dimensionless axial velocity $v_\xi^*(\eta)$ is parabolic in η. The parameter ϵ is defined as

$$\epsilon = D/kh = 1/Sh_w \tag{6}$$

The Peclet number is defined here as $Pe = [h < v_\xi > /D]$ for parallel plates and as $Pe = [4h < v_\xi > /3D]$ for cylindrical tubes.

In the exact solution, many eigenfunctions are required for accuracy at small x. For example, Walker and Davies' solution, with six eigenfunctions, is not adequate for $x < 0.05$. Accordingly, several workers (Colton, 1971; Pancharatnam and Homsy, 1972; Friedman, 1976) have derived entrance-region solutions. These Lévêque types of solutions may be accurate only for $x < 0.01$, as found by Newman (1969) for the cylindrical case with $\epsilon = 0$. We present here an extended entrance-region solution for x up to 0.1 and significant wall resistance ($\epsilon > 1$).

ENTRANCE-REGION SOLUTIONS

In the mass transfer entrance region, the velocity profile is linear within the diffusion layer, and curvature may be neglected. The approximate differential equation for small x becomes

$$3y\frac{\partial \theta}{\partial \bar{x}} = \frac{\partial^2 \theta}{\partial y^2} \tag{7}$$

where

$$y = (1-\eta)/\epsilon \tag{8}$$

and

$$\bar{x} = x/\epsilon^3 \tag{9}$$

Boundary conditions (2) and (4) become

$$\theta = 1 \quad @ \quad \bar{x} = 0 \tag{10}$$

and

$$\frac{\partial \theta}{\partial y} = \theta \quad @ \quad y = 0 \tag{11}$$

The third boundary condition, Equation (3), is approximated by

$$\theta \to 1 \quad \text{at} \quad y \to \infty \tag{12}$$

The solution of this problem may be obtained by Laplace transformation. The wall concentration resulting from inversion is

$$\theta_w(\bar{x}) = \frac{b\sqrt{3}}{2\pi}\int_0^\infty \frac{r^{-2/3}e^{-\bar{x}r}dr}{r^{2/3} + br^{1/3} + b^2} \tag{13}$$

where $b = 0.95109834$. We have evaluated this integral by Romberg integration, and the result is plotted in Figure 1.

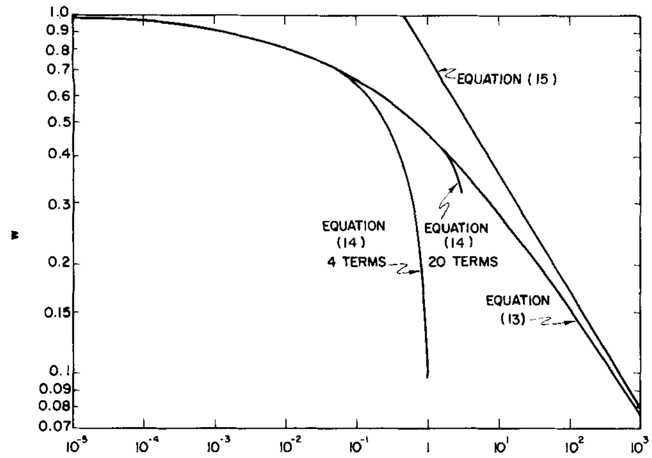

Fig. 1. Wall concentration as a function of $\bar{x}$, the scaled distance from the leading edge of the mass transfer section. Note that the flux is proportional to θ_w.

Pancharatnam and Homsy (1972) have presented another form of the inverse transform, and they and Friedman (1976) give asymptotic forms. For small $\bar{x}$

$$\theta_w(\bar{x}) \sim \sum_{n=0}^{\infty} \frac{(-b\bar{x}^{1/3})^n}{(n/3)!} \tag{14}$$

as obtained by Friedman. Terms of this series through $(\bar{x})^{2/3}$ were given by Colton et al. (1971). Equation (14) appears to diverge at $\bar{x} > 3$ as shown in Figure 1.

The large $\bar{x}$ asymptote obtained by the earlier workers is

$$\theta_w(\bar{x}) \sim \frac{\sqrt{3}}{2\pi} \frac{\Gamma(1/3)}{b} (\bar{x})^{-1/3} + O(\bar{x}^{2/3}) \tag{15}$$

which is also plotted in Figure 1.

EXTENSION TO THICKER DIFFUSION LAYERS

The solutions given above are limited to small values of x primarily by the approximations embodied in (7). By comparison with the Lévêque problem (Newman, 1969), we expect these to fail for $x > 0.01$, whereas Equation (12) should be accurate up to $x = 0.1$.

To estimate the behavior at larger x, we consider Equation (1), including the η^2 term in the velocity, by using the method devised by Newman (1969) for extending the Lévêque solution. That is, we transform the problem by use of the Lévêque similarity variable

$$\beta = (1 - \eta)/x^{1/3} \tag{16}$$

It is also convenient to let

$$z = x^{1/3}/\epsilon = (\bar{x})^{1/3} \tag{17}$$

With these new variables, Equation (1) becomes

$$\left[\beta - \frac{1}{2}\beta^2 z\epsilon\right]\left[-\beta\frac{\partial\theta}{\partial\beta} + z\frac{\partial\theta}{\partial z}\right] = \frac{\partial^2\theta}{\partial\beta^2} \underline{- (z\epsilon + \beta z^2\epsilon^2 + \beta^2 z^3\epsilon^3 + \ldots)\frac{\partial\theta}{\partial\beta}} \tag{18}$$

Boundary condition (4) becomes

$$\frac{\partial\theta}{\partial\beta} = z\theta \quad \text{at} \quad \beta = 0 \tag{19}$$

and Equations (2) and (12) reduce to

$$\theta \to 1 \quad \text{as} \quad \beta \to \infty \tag{20}$$

The underlined terms in Equation (18) arise from the Laplacian in curvilinear coordinates and appear only for the cylindrical geometry.

We now seek a solution of the form

$$\theta(z, \beta; \epsilon) = \theta_0(\beta; \epsilon) + z\theta_1(\beta; \epsilon) + z^2\theta_2(\beta; \epsilon) + \ldots \tag{21}$$

as suggested also by Colton et al. (1971). Substitution of this series into Equations (18), (19), and (20) yields a hierarchy of equations for the functions θ_0, θ_1, θ_2, etc., as follows:

$$\theta_0'' + \beta^2\theta_0' = 0 \tag{22}$$

$$\theta_1'' + \beta^2\theta_1' - \beta\theta_1 = \tfrac{1}{2}\epsilon\beta^3\theta_0' \tag{23}$$

$$\theta_2'' + \beta^2\theta_2' - 2\beta\theta_2 = \tfrac{1}{2}\epsilon\beta^3\theta_1' - \tfrac{1}{2}\epsilon\beta^2\theta_1 + \underline{\epsilon\theta_1} \tag{24}$$

$$\theta_3'' + \beta^2\theta_3' - 3\beta\theta_3 = \tfrac{1}{2}\epsilon\beta^3\theta_2' - \epsilon\beta^2\theta_2 + \underline{\beta\epsilon^2\theta_1' + \epsilon\theta_2'} \tag{25}$$

with

$$\theta_0 = 1 \quad \text{at} \quad \beta \to \infty \quad (\text{and at} \quad z = 0) \tag{26}$$

$$\theta_1 = \theta_2 = \theta_3 = \ldots = 0 \quad \text{at} \quad \beta \to \infty \tag{27}$$

$$\left.\begin{aligned} \theta_1' &= \theta_0 \quad \text{at} \quad \beta = 0 \\ \theta_2' &= \theta_1 \quad \text{at} \quad \beta = 0 \\ \theta_3' &= \theta_2 \quad \text{at} \quad \beta = 0, \quad \text{etc.} \end{aligned}\right\} \tag{28}$$

Again, the underlined terms appear only in the cylindrical case.

Clearly, the first term must be

$$\theta_0 = 1 \tag{29}$$

The solution for θ_1 is

$$\theta_1 = \frac{-3^{1/3}}{\Gamma\left(\frac{2}{3}\right)} \beta \int_\beta^\infty \frac{\exp(-\beta^3/3)}{\beta^2} d\beta \tag{30}$$

a function obtained by Bird (1959) for a constant wall flux. The higher-order terms must be determined numerically. To account for the appearance of ϵ in the nonhomogeneous terms of the differential equations, we write

$$\theta_2(\beta; \epsilon) = f_1(\beta) + f_2(\beta)\epsilon \tag{31}$$

and

$$\theta_3(\beta; \epsilon) = f_3(\beta) + f_4(\beta)\epsilon + f_5(\beta)\epsilon^2 \tag{32}$$

Substitution of Equations (29) to (32) into Equations (24) and (25) yields a system of five coupled differential equations for the f coefficients. We have solved these for both the planar and cylindrical cases by the finite difference method of Newman (1968). It is found that f_1 and f_4 are positive, whereas f_2, f_3, and f_5 are negative. All functions are monotonic and decay to zero by about $\beta = 2$. Therefore, the diffusion layer reaches the center by $x \approx 0.125$.

Once again, we concern ourselves primarily with the concentration and flux distributions at the wall. From the values of the computed functions at $\beta = 0$, namely

Planar	Cylindrical
$f_1^P(0) = 1.002041$	$f_1^C(0) = 1.002041$
$f_2^P(0) = -0.094081$	$f_2^C(0) = -0.6013245$

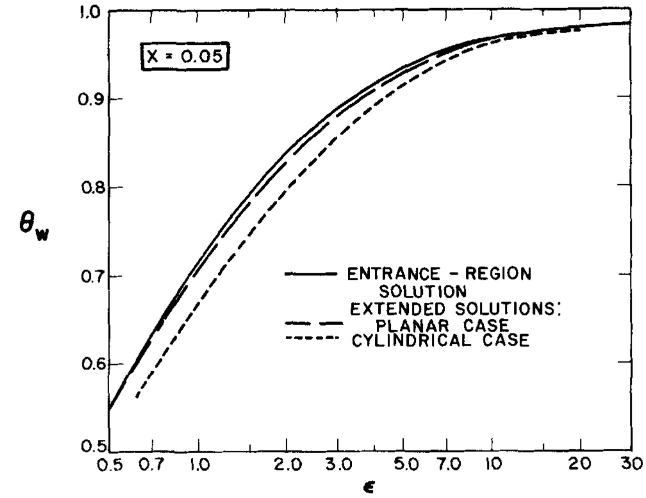

Fig. 2. Wall concentration at $x = 0.05$ as a function of the rate parameter ϵ. The entrance-region solution is Equation (13). The extended solutions include the corrections given in Equations (34) and (35).

$$f_3{}^P(0) = -0.860351 \qquad f_3{}^C(0) = -0.860351$$
$$f_4{}^P(0) = 0.172070 \qquad f_4{}^C(0) = 1.032506$$
$$f_5{}^P(0) = -0.030032 \qquad f_5{}^C(0) = -0.414665$$

we obtain the series

$$\theta_w(\bar{x}; \epsilon) = 1 - 1.065084\,(\bar{x})^{1/3} + 1.002041\,(\bar{x})^{2/3} - 0.860351\,\bar{x} + \Delta + \ldots \quad (33)$$

where

$$\Delta = \Delta^P = -\,[0.094081\,(\bar{x})^{2/3} - 0.172070\,\bar{x}]\epsilon - [0.030032\,\bar{x}]\epsilon^2$$
$$= -\,[0.094081\,x^{2/3} + 0.030032\,x]\epsilon^{-1} + [0.172070\,x]\epsilon^{-2} \quad (34)$$

for the planar case and

$$\Delta = \Delta^C = -\,[0.601324\,(\bar{x})^{2/3} - 1.032506\,\bar{x}]\epsilon - [0.414665\,\bar{x}]\epsilon^2$$
$$= -\,[0.601324\,x^{2/3} + 0.414665\,x]\epsilon^{-1} + [1.032406\,x]\epsilon^{-2} \quad (35)$$

for the cylinder.

It is interesting to note that the coefficients in Equation (33) are the same as those generated by Equation (14). Therefore, the functions Δ^P and Δ^C given in Equations (34) and (35) may be viewed as large x, finite ϵ corrections to the small $\bar{x}$ solutions.

Figure 2 shows the wall concentration as a function of ϵ at $x = 0.05$. The solid curve is the entrance-region solution obtained in Equation (13). The two dashed curves are the values corrected by Equations (34) and (35) for the planar and cylindrical geometries, respectively. It is seen that the corrections are small, even at this x, where the eigenfunction solutions become useful. The correction for the cylindrical tube is much greater than for the parallel plates, which indicates that the curvature in the diffusion layer is more important than the nonlinearity of the velocity profile.

The extended solution obtained here is of the same form as Equation (14), so that its region of convergence must be similar, that is, $\bar{x} < 3$ or $\epsilon > (x/3)^{1/3}$. Accordingly, the results are limited to large ϵ at finite x. At $\epsilon = 1$, $x = 0.05$, θ_w from Equation (13) is 0.7105. The value corrected by Equation (34) is 0.7047. The value given by Walker and Davies (1974) is 0.7018. The correction is beginning to diverge here.

For the cylindrical case at $\epsilon = 0$, Newman (1969) found the Lévêque solution to err by 24% in the total mass transferred within $0 < x < 0.05$. Thus, for small ϵ the corrections should be greater than those calculated here. In most dialyzers, however, ϵ is greater than 1 (Cooney et al., 1974), so that in such applications the Lévêque types of entrance solutions are adequate to supplement the eigenfunction solution at small x.

NOTATION

D = diffusion coefficient (L^2t^{-1})
h = channel half width or tube radius (L)
k = first-order reaction rate constant or mass transfer coefficient for transport through the wall (Lt^{-1})
Pe = Peclet number, defined after Equation (6)
Sh_w = wall Sherwood number
$<v_\xi>$ = mean axial velocity at distance ξ downstream (Lt^{-1})
$v_\xi{}^*$ = dimensionless axial velocity = $v_\xi/<v_\xi>$
x = dimensionless axial coordinate = $(\xi/h)/Pe$
$\bar{x}$ = scaled dimensionless axial coordinate = x/ϵ^3
y = scaled dimensionless distance from wall = $(1 - \eta)/\epsilon$
z = transformed dimensionless axial coordinate = $(\bar{x})^{1/3}$

Greek Letters

β = Lévêque similarity variable = $(1 - \eta)/x^{1/3}$
Γ = gamma function
Δ = correction function defined by Equation (34) or (35)
ϵ = D/hk, dimensionless rate parameter
η = dimensionless transverse coordinate measured from center plane or axis
θ = dimensionless concentration
ξ = axial coordinate (L)

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